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NOTE

# CUBE TILING AND COVERING A COMPLETE GRAPH

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A collection of translates of an $n$-dimensional cube forms a *tiling* if the $n$-space is covered by its elements and no two cubes have a common interior point. If two $n$-dimensional cubes share a whole common $(n-1)$-dimensional face then they will be called a *twin pair*.

In 1930 Keller [2] conjectured that in any $n$-dimensional cube tiling there is a twin pair. In 1940 Perron [5] proved this conjecture for $n \leqslant 6$, but it is still unsettled for $n \geqslant 7$. In 1982 Lawrence [3], using geometrical argumentation, gave the following graph theoretical form for Keller's conjecture. Let $G_1, \ldots, G_n$ and $K$ be bipartite graphs and the complete graph of the same vertex set $V$ of $2^n$ vertices respectively. If $G_1 \cup \cdots \cup G_n = K$ then there exists an edge $e$ of $G_i$ such that $e \cup G_j$ contains an odd cycle for each $j$, $1 \leqslant j \leqslant n$ and $j \neq i$.

In this note we derive a slightly different graph test for Keller's conjecture from its group theoretical version.

**Theorem.** *Let $G_1, \ldots, G_n$ be bipartite graphs of the same vertex set $V$ such that each of them is a disjoint union of at most two regular complete bipartite graphs. If $G_1 \cup \cdots \cup G_n$ is the complete graph of vertex set $V$ then there exists an edge $e$ of $G_i$ such that $e \cup G_j$ contains a triangle for each $j$, $1 \leqslant j \leqslant n$, $j \neq i$.*

**Proof.** In the remaining part we sketch the proof. Let $H$ be an additive abelian group of order $4^n$ with basis elements $h_1, \ldots, h_n$ of order 4 and let $L$ be a subset of $2^n$ elements of $H$. According to [8] Keller's conjecture is equivalent to the following implication. If

$$H = L + \{0, h_1\} + \cdots + \{0, h_n\} \tag{1}$$

then

$$(L-L)\cap\{2h_1,\ldots,2h_n\}\neq\emptyset, \tag{2}$$

where

$$L-L=\{l-l'\colon\ \ l,l'\in L\}.$$

To derive the new graph test let

$$M=\{0,h_1\}+\cdots+\{0,h_n\}.$$

f $h\in H$ and

$$h=x_1h_1+\cdots+x_nh_n,\quad 0\leqslant x_1,\ldots,x_n\leqslant 3$$

then we refer to $x_i$ as the $i$th basis coefficient of $h$.

Clearly, $|M|=2^n$ therefore $|H|=|L|\,|M|$; and consequently each $h\in H$ is uniquely expressible in the form

$$h=l+m,\quad l\in L, m\in M.$$

Here $|A|$ is the number of elements of $A$. For fixed $i$, $1\leqslant i\leqslant n$ the elements of $L$ can be divided into four disjoint classes $L_{i0}$, $L_{i1}$, $L_{i2}$, $L_{i3}$. If the $i$th basis coefficient of $l\in L$ is $k$ then let $l$ belong to the class $L_{ik}$. We will use the fact that

$$|L_{i0}|=|L_{i2}|,\qquad |L_{i1}|=|L_{i3}|. \tag{3}$$

To prove it consider the character $\chi$ of $H$ defined by

$$\chi(h_i)=\rho\quad\text{and}\quad\chi(h_j)=1\quad\text{for } j\neq i,$$

where $\rho$ is a fourth primitive root of unity. Note that $\chi$ is not the principal character of $H$ hence

$$0=\sum_{h\in H}\chi(h)=\Big(\sum_{l\in L}\chi(l)\Big)\Big(\sum_{m\in M}\chi(m)\Big)$$

and

$$0\neq\sum_{m\in H}\chi(m)$$

implies that

$$0=\sum_{l\in L}\chi(l)=|L_{i0}|\rho^0+|L_{il}|\rho^1+|L_{i2}|\rho^2+|L_{i3}|\rho^3$$

which proves (3).

Note that (1) is equivalent to

$$(L-L)\cap(M-M)=\{0\}. \tag{4}$$

The basis coefficients of $m\in M$ are 0 and 1 so $M-M$ consists of all elements of $H$ whose basis coefficients are 0, 1 and 3.

Now we define graphs $G_1,\ldots,G_n$ on the vertex set $L$. Let $l,l'\in L$ and $l\neq l'$. If the $i$th basis coefficient of $l-l'$ is 2 then let $(l,l')$ be an edge of $G_i$. $G_i$ is the union of the complete bipartite graphs of the vertex set pairs $(L_{i0},L_{i2})$ and

$(L_{i1}, L_{i3})$ respectively. Further $G_1 \cup \cdots \cup G_n$ is the complete graph of the vertex set $L$ since according to (4) for each $l, l' \in L$ and $l \neq l'$ one of the basis coefficients of $l - l'$ must be 2. If $G_1, \ldots, G_n$ is given then we can construct the subsets $L_{i0}$, $L_{i1}$, $L_{i2}$, $L_{i3}$ and through them the $i$th basis coefficients of elements of $L$. Since $M$ is known $H$ has an expression in form (1).

The graph theoretical formulation of (2) is now clear. Let $l, l' \in L$, $l \neq l'$ such that $l - l' = 2h_i$. For fixed $j$, $j \neq i$, $l$ and $l'$ are in the same $L_{jk}$ set, say $l, l' \in L_{j0}$. According to (3) $L_{j2} \neq \emptyset$ and so there exists $l'' \in L_{j2}$ such that $(l, l'')$ and $(l'', l')$ are edges of $G_j$. Finally, assume that there exists an edge $(l, l')$ of $G_i$ such that $(l, l') \cup G_j$ contains a triangle $(l, l')$, $(l', l'')$, $(l'', l)$ for each $j$, $1 \leq j \leq n$, $j \neq 1$. Now $l$ and $l'$ belong to the same $L_{jk}$ set and so the $j$th basis coefficient of $l - l'$ is zero. But since $(l, l')$ is an edge of $G_i$ the $i$th basis coefficient of $l - l'$ must be 2 and therefore $l - l' = 2h_i$. $\square$

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